The "Optimum" Putt

... on a slanted and stepped green

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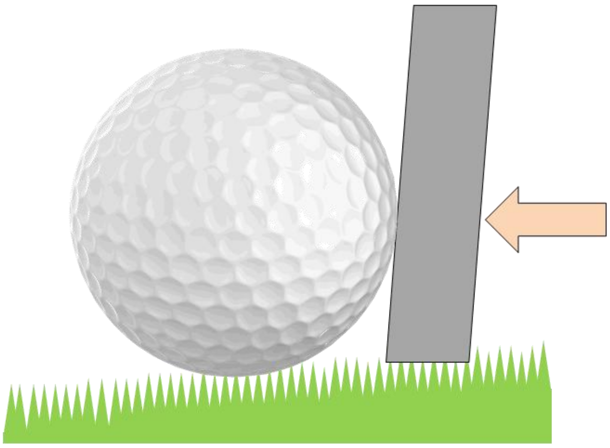

Abstract

The movement of a golf ball is from a physical and mathematical point of view the result of a solution of a differential equation under consideration of all physical parameters on a slanted and/ or stepped green. For practical golfers it is nevertheless a huge problem to identify the important characteristics of a moving ball on a green with various parameters. Golf player are spending many hours on a green to identify and practice "The Optimum Putt" in all circumstances which is in reality not easy.

In this contribution we will evaluate on the basics of an ideal friction free slanted green the principals of green reading and introduce the aim point for the practical golfer. We will solve the exact solution of Newton's equation of a rolling uphill as well as downhill ball and will evaluate practical approximations for the aim point.

Based on this exact calculations we are able to compare this result with other proposed solutions.

The final application of the exact solution is the stepped green, where approximations for uphill and downhill putts are presented.

With the knowledge of above mentioned approximations every golfer has now at least a better chance to hit the famous hole - because he know now the exact aim point. But nevertheless he has still to hit the ball with the "right" velocity which he will learn as usual only with daily training on a putting green.

Index Terms

golf, putt, slanted green, stepped green, aim point

I. Introduction

A Putter with a typical loft angle of 4 degree will hit the golf ball with a radius of 21,3 mm and a weight of 46 gram slightly below the equator with a certain velocity

1. the ball will start with a minor inclined pitch for some centimetre before
2. continue to glide on the green for some decimetre
3. until the ball will start rolling several metre on the green and will hopefully fall down into the golf hole

Every golf player is highly interested to sink the ball at the famous hole as early as possible. We will spend most of the paper for answering the question what is "The Optimum Putt" . We will concentrate on the question what are the decisive factors in respect to golf ball velocity in the right direction under the various unknown factors on a golf green.

The solutions for the ball movement on a slanted green or a stepped green are in reality highly complex. For practical golfer we will present easy approximations for the most important parameter - the aim point - which is the preferred direction of the golf ball hit. On this basis a golfer has just to play the ball with the "right" velocity in this direction to the "aim point" to sink the ball in the hole.

The concept of hitting a ball into the direction of this aim point is well known in the golf world (see reference 1... 4). But with the exact solution of Newton's equation of a rolling ball on a slanted surface with a given fiction we are able to establish a physical and mathematical correct solution to the never ending question of "The Optimum Putt".

Beside the interesting mathematical derivation which will be described in detail please have a special look to the approximations for the aim point.

- putt on a slanted green in para 5, equation 5.18/19
- and for a stepped green in para 7, equation 7.4/ 5

II. Some basics for jumping and gliding

Every putt starts with a small jump. The coordinates of a point target will follow the simple rule of inclined pitch as follow:

$$y = V_0\, t\, \sin\varepsilon - \frac{1}{2} g\, t^2$$

$$x = V_0\, t\, \cos\varepsilon$$

with ball velocity V_0 and acceleration of gravity $g = 9{,}81 \frac{m}{s^2}$ *or* $32{,}185\ \frac{feet}{s^2}$ and loft angle of putter ε

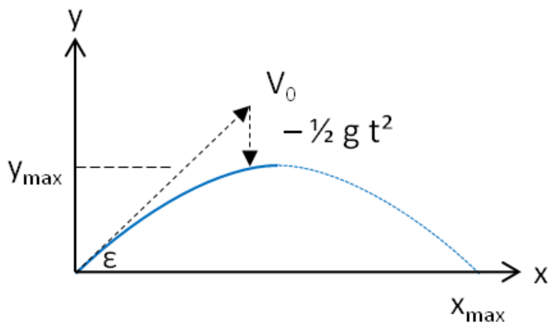

Fig. 1 - slanted pitch

The motion equation of a jumping golf ball is simply

$$y = x \tan\varepsilon - \frac{g\, x^2}{2\,(v_0\, \cos\varepsilon)^2}$$

The maximum distance in x-direction will be reached for $y = 0$, therefore

$$x_{max} = \frac{2\, {v_0}^2 \sin\epsilon\, \cos\epsilon}{g} \qquad (1.1)$$

For example: with $v_0 = 2\frac{m}{s}$ and $\epsilon = 4\ degree$ the maximum distance of the minor pitch is only 56 mm (no wonder that nearly nobody realize this, because the ball itself has already a diameter of 42,6 mm)

This small jump will be followed by horizontal gliding on the green with a gliding friction coefficient μ_g

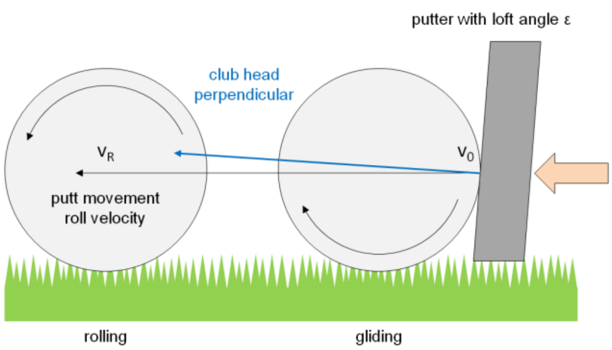

Fig. 2 - from gliding to rolling

As usual we will start with Newton's basics

$$F = m\, a = m\, \frac{dv}{dt} = -\mu_g\, m\, g$$

where the force F is equal to the mass m of the ball multiplied with the acceleration a. Integration over time leads to velocity v and distance s_g during gliding

$$v = v_0 - \mu_g\, g\, t \text{ and to}$$

$$s_g = v_0 t - \frac{1}{2}\mu_g g\, t^2$$

By the way the friction force coefficient of a gliding ball on a green is in the order of $\mu_g = 0{,}2 \ldots 0{,}4$.

The gliding golf ball with radius $r = 21{,}3\ mm$ has a mass moment of

$$M = J\, \alpha = \frac{2}{5}\, m\, r^2\, \frac{d\omega}{dt} = \mu_g m\, g\, r$$

where J is the moment of inertia of a sphere and ω is the angular speed. Integration of ω over time lead to

$$\omega = \frac{5\, \mu_g g}{2\, r}\, t$$

At time t the radial velocity of the golf ball will reach

$$v = \omega\, r = \frac{5\, \mu_g g}{2\, r}\, t\, r = v_0 - \mu_g\, gt$$

The golf ball start rolling after $\quad t_r = \frac{2}{7}\frac{v_0}{\mu_g g}$

And the gliding distance will be $\quad s_g = \frac{12}{49}\frac{{v_0}^2}{\mu_g g}$

For example: with $v_0 = 2\frac{m}{s}$ and glide riding friction coefficient $\mu_g = 0{,}4$ the gliding distance is 0,249 m.

Edward Stimpson has established in 1935 with an equivalent equation for a rolling ball a standard for green measurement speed. With a standard roll velocity v_r of 6 feet/sec the measured rolling distance s_r in feet is the so called Stimp factor SF which is based on a rolling friction coefficient μ_r

$$\mu_r = \frac{1}{2}\frac{{v_r}^2}{s_r g} = \frac{0{,}559}{SF} \qquad (1.2)$$

which reach values between 0,14 for slow greens with SF = 4 and 0,056 for very fast greens of SF = 10 for professional golfers on PGA tours worldwide.

III: The slanted green

For all calculations we will use the following system of coordinates

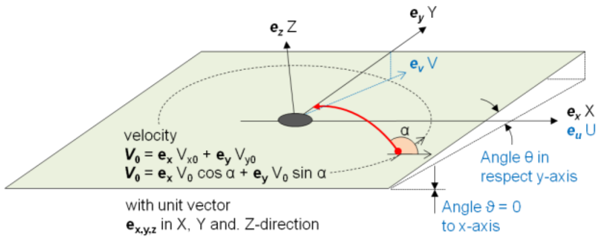

Fig. 3 - slanted green and system of coordinates

The golf ball will start somewhere on the slanted green with a certain velocity V_0 in the direction α. The green itself has a slant angle Θ in respect to the y-axis.

A point P(x,y,z) on the slanted green has the following relationship with the horizontal u,v,w system

$$P(X,Y,Z) = e_x X + e_y Y + e_z Z$$

$$X = U$$

$$Y = \quad V \cos\Theta + W \sin\Theta$$

$$Z = -V \sin\Theta + W \cos\Theta$$

The moving ball in x,y plane will be influenced by a friction force which is proportional to the mass of the ball and the gravity force in W axis.

IV: A perfect green without friction

We will start with an ideal green without friction to establish some basic understanding of putting principles for future use.

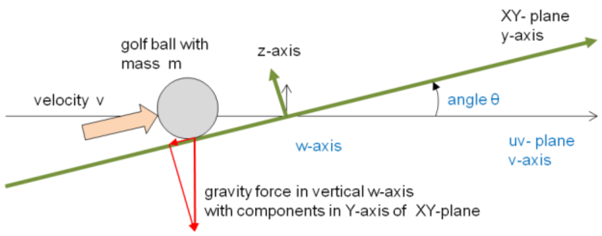

Fig. 4 - slanted XY-plane

Newton's equation of motion in x-axis is very easy, just

$$m \frac{d^2x}{dt^2} = 0 \qquad \text{no friction at all in x-direction}$$

Integration of this equation with start condition at $t = 0$ for velocity v_0 and distance x_0 leads to

$$\int \frac{d^2x}{dt^2}\,dt = constant = v_{x0}$$

$$\int \frac{dx}{dt} dt = x = v_{x0}t + contant = v_{x0}t + x_0$$

which leads to $\qquad t = \frac{x - x_0}{v_{x0}}$

Similar the equation for ball movement in y-axis, but now under consideration of gravity contribution in y-direction

$$m \frac{d^2y}{dt^2} = -m\, g\, \sin\Theta$$

$$\int \frac{d^2y}{dt^2}\,dt = -\int g\,\sin\Theta\, dt = -g\sin\Theta\; t + v_{y0}$$

$$\int \frac{dy}{dt}\,dt = \int (-g\sin\Theta\; t + v_{y0})dt$$

$$y = -\frac{1}{2}\, g\,\sin\Theta\, t^2 + v_{y0}t + y_0$$

Use of time relationship leads to the final solution of a parabola of golf ball movement. (4.1)

$$y = -\frac{1}{2}\, g \left[\frac{x - x_0}{v_{xo}}\right]^2 \sin\Theta + v_{y0}\left[\frac{x - x_0}{v_{x0}}\right] + y_0$$

The slope of the parabola, the tangent will be the derivative of y

$$\frac{dy}{dx} = -\frac{1}{2}\frac{2\,(x - x_0)}{(v_{x0})^2}\, g\,\sin\Theta + v_{y0}\,\frac{1}{v_{x0}}$$

$$\frac{dy}{dx} = \frac{v_{y0}}{v_{x0}} - (x - x_0)\frac{g\,\sin\Theta}{(v_{x0})^2}$$

The point of intersection of this tangent at point $x_0; y_0$ with the y-axis at x=0 is the so called "aim point"

$$y_{aim\ point} = -\frac{v_{y0}}{v_{x0}}\, x_0 + y_0 \qquad (4.2)$$

The putting hole is located at $x = y = 0$. For an uphill putt from the lower part of the green to the upper part the velocity in y-direction at the hole has to be zero.

$$\frac{dy}{dt} = v_{y\ hole} = 0 = -t\, g\,\sin\Theta + v_{y0}$$

Therefore the time of rolling $\qquad t = \frac{v_{y0}}{g\,\sin\Theta}$

Plug in the t-relationship of x $\qquad v_{x0} = -\frac{x_0\, g\,\sin\Theta}{v_{y0}}$

We would like to sink the golf ball at $y = 0$

$$y = 0 = -\frac{1}{2}\left[\frac{-x_0}{v_{x0}}\right]^2 g\ \sin\Theta + v_{y0}\left[\frac{-x_0}{v_{x0}}\right] + y_0$$

With above calculated v_{x0} we will reach the following equation for the optimum speed in y-direction

$$v_{y0} = \pm\sqrt{y_0\ 2\ g\ \sin\Theta} \qquad (4.3)$$

and for the optimum speed in x-direction

$$v_{x0} = \mp\frac{x_0\ g\ \sin\Theta}{\sqrt{|y_0\ 2\ g\ \sin\Theta|}} \qquad (4.4)$$

Using both results for the velocity components in equation (4.2) for the aim point leads to

$$\frac{v_{y0}}{v_{x0}} = 2\ \frac{y_0}{x_0}$$

The point of intersection between tangent of the parabola with the y-axis at x=0 is the so called "aim point"

$$y = \frac{dy}{dx}\,(x - x_0) + y_0 = 2\ \frac{y_0}{x_0}\,(-x_0) + y_0$$

$$y_{aim\ point} = -y_0$$

Surprisingly any uphill putt will use as the "Aim Point" the opposite side of y_0 independent of its x_0-position.

A downhill putt from the upper plane to the centre hole at $x = y = 0$ has to reach the centre with minimum velocity in x-direction (in y-direction the force of gravity will accelerate anyhow the ball).

$$y = 0 = -\frac{1}{2}\ g\left[\frac{-x_0}{v_{xo}}\right]^2 \sin\Theta + v_{y0}\left[\frac{-x_0}{v_{x0}}\right] + y_0$$

$$v_{y0} = \frac{v_{x0}}{x_0}\left[y_0 - \frac{1}{2}\left(\frac{x_0}{v_{x0}}\right)^2 g\ \sin\Theta\right]$$

The optimum velocity is the result of min-max equation

$$\frac{dv_{y0}}{dv_{x0}} = \frac{y_0}{x_0} - \frac{x_0}{(v_{x0})^2}\,\frac{g\ \sin\Theta}{2} = 0$$

$$v_{x0} = \pm\sqrt{\frac{(x_0)^2}{|y_0|}\,\frac{g\ \sin\Theta}{2}} \qquad (4.5)$$

$$v_{y0} = 0 \qquad (4.6)$$

Any downhill putt has to start horizontal !

The following picture will illustrate the results of an optimum putt on an ideal green without any friction to the centred hole on the slanted x-y plane.

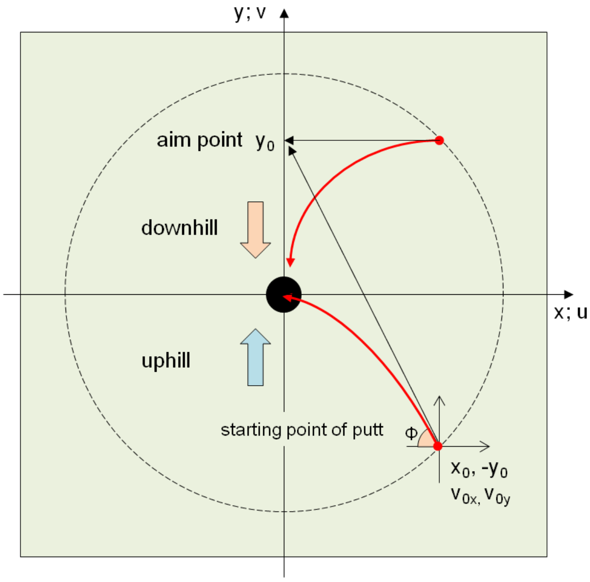

Fig. 5 - putt illustration for an ideal friction free surface

An Optimum Putt on an ideal green without any rolling friction has to direct all putts to the "Aim Point" y_0 above the centre hole, independent of the x-position !

See also a practical graphics of uphill and downhill putt.

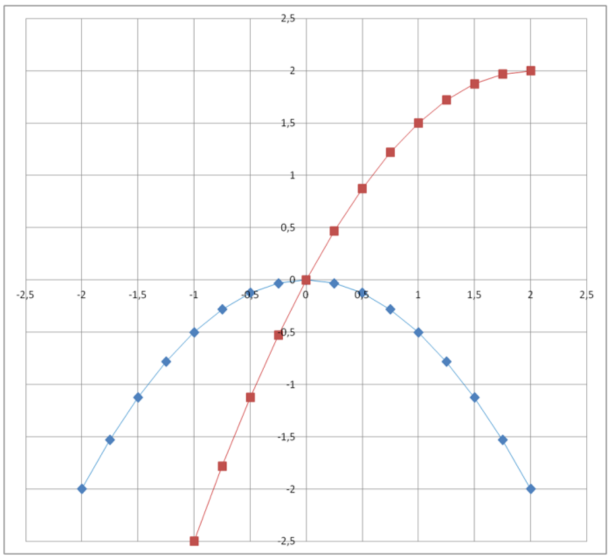

Fig. 6 - putt lines of optimum putts towards the centre

Putt parameter:

Uphill	2,00	$x_0/meter$	2,00	Downhill
	-2,00	$y_0/meter$	2,00	
	-0,72	v_{x0} m/sec	-0,72	
	1,43	v_{y0} m/sec	0	
	3	Θ /degree	3	

V: A slanted green with friction

A rolling golf ball will be influenced by two forces

1. Gravity force which will accelerate for downhill situation or stop the ball for uphill movement
2. Friction force of rolling with a friction coefficient μ_R which is proportional to the weight of golf ball. This force is acting against the normal movement

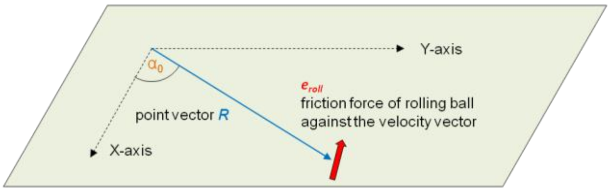

Fig. 7 - rolling ball on a slanted green with friction

Newton's equation for the point vector R *will be*

$$m\,\frac{d^2R}{dt^2} = e_x m\,g\,\sin\Theta - e_{roll}\,\mu_R\,m\,g\,\cos\Theta \qquad (5.1)$$

with $\qquad e_{roll} = e_x\cos\alpha + e_y\,\sin\alpha$

and $\qquad M = \mu_R\,\cot\Theta \qquad$ as abbreviation

we will get the ball movement in X-Y coordinates (5.2)

$$m\,\frac{d^2R}{dt^2} = m\,g\,\sin\Theta\,\{e_x\,(1 - M\cos\alpha) - e_y\,M\,\sin\alpha\}$$

The tangent component T of vector R is $T = R \times e_{roll}$

$$m\,\frac{d^2T}{dt^2} = m\,g\,\sin\Theta\,\{(1 - M\cos\alpha)\cos\alpha - M\,sin^2\alpha\}$$

$$m\,\frac{d^2T}{dt^2} = m\,g\,\sin\Theta\,\{\cos\alpha - M\}$$

Sum up of X-component and T-component leads to

$$m\,\frac{d^2X}{dt^2} + M\,m\,\frac{d^2T}{dt^2} = \ldots$$

$$= mg\sin\Theta\,(1 - M\cos\alpha) + M\,mg\sin\Theta\,\{\cos\alpha - M\}$$

$$m\,\frac{d^2X}{dt^2} + M\,m\,\frac{d^2T}{dt^2} = mg\,\sin\Theta\,[1 - M^2]$$

This means, that the sum of X and T components are surprisingly constant and could be easily integrated

$$\frac{dX}{dt} + M\,\frac{dT}{dt} = mg\,\sin\Theta\,[1 - M^2]\,t + constant$$

At time t=0 the ball start with a certain velocity V_0 in the direction of angle α_0, therefore the components

$$constant = V_0\,\cos\alpha_0 + M\,V_0$$

$$V\cos\alpha + MV = g\,\sin\Theta\,[1 - M^2]t + V_0[M + \cos\alpha_0]$$

Which will easily lead to an equation for the velocity V

$$V = \frac{g\sin\Theta(1-M^2)t + V_0(M+\cos\alpha_0)}{M+\cos\alpha} \qquad (5.3)$$

and the velocity components in x and y direction are

$$V_x = V\,\cos\alpha \qquad (5.4)$$

$$V_x = V\,\sin\alpha \qquad (5.5)$$

The velocity components in x and y axis are arranged in angle α

$$\tan\alpha = \frac{V_y}{V_x} = \frac{dY/_{dt}}{dX/_{dt}}$$

$$\alpha = \tan^{-1}\frac{dY/_{dt}}{dX/_{dt}}$$

Derivation lead to an angle velocity $\frac{d\alpha}{dt}$

$$\frac{d\alpha}{dt} = \frac{1}{1 + \left[\frac{dY/_{dt}}{dX/_{dt}}\right]^2}\,\frac{\frac{d^2Y}{dt^2}\frac{dX}{dt} - \frac{d^2X}{dt^2}\frac{dY}{dt}}{\left(\frac{dX}{dt}\right)^2}$$

$$\frac{d\alpha}{dt} = \frac{1}{1 + [\tan\alpha]^2}\,\frac{\frac{d^2Y}{dt^2} - \frac{d^2X}{dt^2}\frac{dY}{dt}\Big/\frac{dX}{dt}}{\frac{dX}{dt}}$$

with 5.3 … 5.5 we have the basis for a relationship of $\frac{d\alpha}{dt}$

$$\frac{d\alpha}{dt} = \frac{-g\,\sin\Theta\sin\alpha\,(M+\cos\alpha)}{g\,\sin\Theta\,(1-M^2)t + V_0(M+\cos\alpha_0)} \qquad (5.6)$$

The variable α and t are easily to separate which is the basis for integration of both sides, multiplication of both sides with $g\sin\Theta\,(1 - M^2)$ leads to

$$\int\frac{-g\sin\Theta(1-M^2)\,d\alpha}{g\,\sin\Theta\sin\alpha\,(M+\cos\alpha)} = LS \qquad (5.7)$$

$$RS = \int\frac{g\sin\Theta\,(1 - M^2)\,dt}{g\,\sin\Theta\,(1 - M^2)t + V_0(M + \cos\alpha_0)}$$

The right side is a standard integral of rational function

$$\int \frac{a\,dt}{at+b} = LN\,[g\ \sin\Theta\,(1-M^2)t + V_0(M+\cos\alpha_0)]$$

which results within the limits t = 0 until t

$$\int_{t=0}^{t} \dots dt = LN\left[\frac{g\ \sin\Theta(1-M^2)t+V_0(M+\cos\alpha_0)}{V_0(M+\cos\alpha_0)}\right] \quad (5.8)$$

The left side 5.7 of the integral is a rational fraction of trigonometric functions which will be solved by a partial fraction expansion

$$\int \frac{-(1-M^2)d\alpha}{(M+\cos\alpha)\sin\alpha} = \int \frac{-(1-M^2)\sin\alpha\, d\alpha}{(M+\cos\alpha)\,(1-\cos^2\alpha)}$$

substitute $u = \cos\alpha$ and $du = \sin\alpha\; d\alpha$ results in

$$\int \dots d\alpha = \int \frac{(1-M^2)du}{(M+u)(1-u^2)}$$

$$\int \dots d\alpha = (M^2-1)\int\left[\frac{A\,du}{(u+M)} + \frac{B\,du}{(u+1)} + \frac{C\,du}{(u-1)}\right]$$

Coefficient comparison leads to constants A, B, C

$$C = \frac{(M-1)}{2(M^2-1)} \qquad B = \frac{-(M+1)}{2(M^2-1)} \qquad A = \frac{1}{(M^2-1)}$$

We have now to solve

$$\int \dots d\alpha = \int\left[\frac{du}{(u+M)} + \frac{(1+M)\,du}{2\,(1+u)} + \frac{(1-M)\,du}{2\,(1-u)}\right]$$

which are again rational function as before, therefore

$$\int \dots d\alpha = ln(u+M) - 0{,}5(1+M)\,ln(1+u) + 0{,}5(1-M)\,ln(1-u)$$

$$\int \dots d\alpha = \ln\left[\frac{(M+\cos\alpha)\sqrt{(1-\cos\alpha)^M}}{\sqrt{(1+\cos\alpha)}\ \sqrt{(1-\cos\alpha)}\ \sqrt{(1+\cos\alpha)^M}}\right]$$

Please note that the first two square roots will be $\sin\alpha$ and the other two square roots are $\tan\alpha/2)$

Within the border limits from α_0 to α we got finally

$$\int_{\alpha_0}^{\alpha} \dots d\alpha = \ln\frac{(M+\cos\alpha)\sin\alpha_0\ \ (\tan\alpha/2)^M}{\sin\alpha\ (M+\cos\alpha_0)(\tan\alpha_0/2)^M} \quad (5.9)$$

With 5.7 ... 5.8 we got finally a parametric solution for the time as follows: (5.10)

$$t = \frac{V_0(M+\cos\alpha_0)}{g\sin\Theta\,(1-M^2)}\left[\frac{(M+\cos\alpha)\sin\alpha_0\ \ (\tan\alpha/2)^M}{\sin\alpha\ \ (M+\cos\alpha_0)(\tan\alpha_0/2)^M} - 1\right]$$

Use of 5.10 will lead to velocities in x and y direction

$$V_x = V_0\cos\alpha\,\frac{(M+\cos\alpha_0)}{(M+\cos\alpha)}\,[\dots] \quad (5.11)$$

$$V_y = V_0\sin\alpha\,\frac{(M+\cos\alpha_0)}{(M+\cos\alpha)}\,[\dots] \quad (5.12)$$

whit bracket $\quad [\dots] = \frac{(M+\cos\alpha)\sin\alpha_0\ \ (\tan\alpha/2)^M}{\sin\alpha\ (M+\cos\alpha_0)(\tan\alpha_0/2)^M} \quad (5.13)$

The parametric velocity vectors in x and y direction are related with time

$$\frac{dX}{dt} = \frac{dX\,(\alpha)}{d\alpha}\,\frac{d\alpha(t)}{dt}\ >>>.\ \ \frac{dX\,(\alpha)}{d\alpha} = \frac{dX}{dt}\Big/\frac{d\alpha(t)}{dt}$$

We need to calculate the derivation of α over time from equation 5.6 and 5.10 as follows

$$\frac{d\alpha}{dt} = -\frac{g\ \sin\Theta\ \sin\alpha\,(M+\cos\alpha)}{V_0(M+\cos\alpha_0)[\dots]}$$

With equation 5.11 for x-component and similar 5.12 for y-component

$$\frac{dX(\alpha)}{d\alpha} = -\cos\alpha\,\frac{(V_0)^2\,(M+\cos\alpha_0)^2}{g\sin\Theta\sin\alpha\,(M+\cos\alpha)^2}[\dots]^2$$

$$\frac{dY(\alpha)}{d\alpha} = -\sin\alpha\,\frac{(V_0)^2\,(M+\cos\alpha_0)^2}{g\sin\Theta\sin\alpha\,(M+\cos\alpha)^2}[\dots]^2$$

Use the universal substitution in $u = \tan\frac{\alpha}{2}$ for solving these two integrals for y position of the ball (5.16)

$$y(\alpha) = \frac{(V_0)^2\,sin^2\alpha_0}{2\ g\ \sin\Theta\,(\tan\frac{\alpha_0}{2})^{2M}}\left[\frac{(\tan\frac{\alpha}{2})^{2M-1}}{2M-1} + \frac{(\tan\frac{\alpha}{2})^{2M+1}}{2M+1} - \frac{(\tan\frac{\alpha_0}{2})^{2M-1}}{2M-1} - \frac{(\tan\frac{\alpha_0}{2})^{2M+1}}{2M+1}\right]$$

And similar for the x-position (5.17)

$$x(\alpha) = \frac{(V_0)^2\,sin^2\alpha_0}{4\ g\ \sin\Theta\,(\tan\frac{\alpha_0}{2})^{2M}}\left[\frac{(\tan\frac{\alpha}{2})^{2M-2}}{2M-2} + \frac{(\tan\frac{\alpha}{2})^{2M+2}}{2M+2} - \frac{(\tan\frac{\alpha_0}{2})^{2M-2}}{2M-2} - \frac{(\tan\frac{\alpha_0}{2})^{2M+2}}{2M+2}\right]$$

In this respect we have now an exact solution for the ball movement on a slanted green. We will use this results for the detailed analysis of various cases and will develop on this basis practical approximations.

See the following example of a downhill putt with different start conditions.

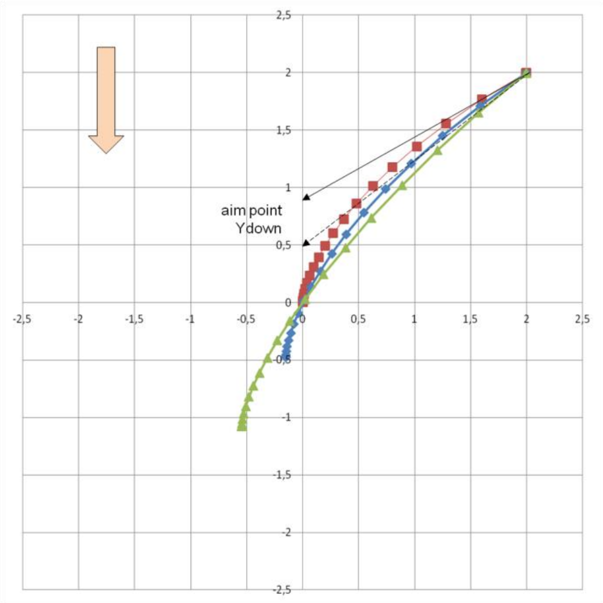

Fig. 8 - $x = y = 2m, \Theta = 2$ degree, $\mu_R = 0{,}007$

You see the different ball lines for

- direct putt with zero velocity at the hole
- a putt which will stop at least 0,5 m at the rear of the hole (in case you should miss the hole !)
- a central putt with 1 m/sec velocity which will stop (in case of missing the hole) by ~1,2 m.

Please note the different aim points on the y-axis between 0,46 m and 0,91 m for start velocities between 1,62 m/sec ... 2,26 m/sec.

A similar example for an uphill putt.

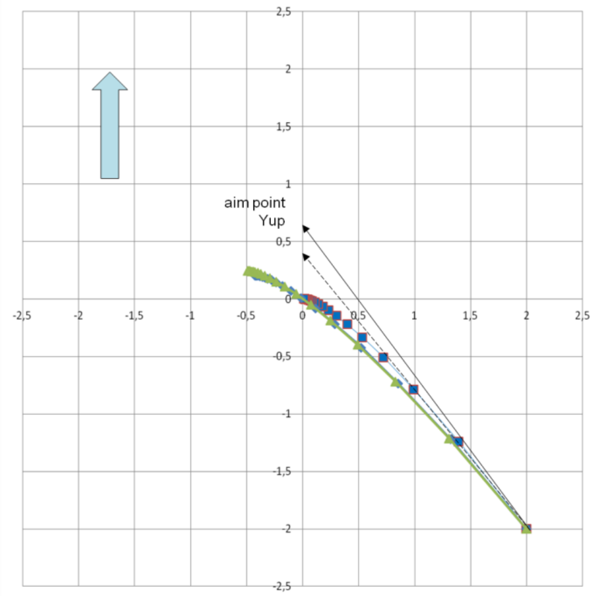

Fig. 9 - $x = 2m, y = -2m, \Theta = 2$ degree, $\mu_R = 0{,}007$

The aim point will be in a smaller region between 0,37 m and 0,61 m for velocities between 2,3 and 2,5 m/s.

The downhill putt is more risky than an uphill putt (as every golfer know). The above mentioned variations for the ball velocity is essential in your real performance as golfer.

By the way: 10 % more speed will extend the rolling distance also by roughly 10 percent. You have to train yourself to hit the ball with different and controlled speed.

The optimum downhill/ uphill putt

The optimum putt strategy is dependent from the correct angle/ aim point at y-axis and the perfect velocity. A slow moving ball will be strongly influenced by some unevenness of the green, a fast ball will stop at a greater distance - if you miss the hole. Most probably a ball with a medium velocity between the two extremes is the optimum solution. For that reason the putt which will go in case of missing the hole just a certain distance to the rear of the hole (for example 0,5 m) will be best

The following diagram demonstrates for the critical downhill putt and different slant angles the aim point.

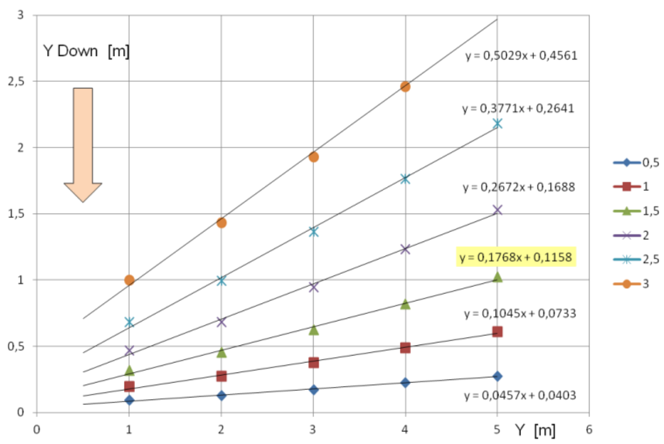

Fig. 10 - Downhill putts with "0,5m putt strategy" for different slant angles Θ of the green.

Carefully analysis of this chart indicates a quadratic dependence of aim point by Θ. Based on this result we have evaluated the following approximation

$$Y \text{ aim point} \sim 11 \text{ mm SF } \theta^2 \text{ Y} \qquad (5.18)$$

with	SF = Stimpfactor of green		6
	Θ = slant angle in degree		2
	Y = distance in meter		3

Example: Y = 11mm 6 2² 3 = 792 mm (see above)

A similar diagram has been calculated for the uphill case

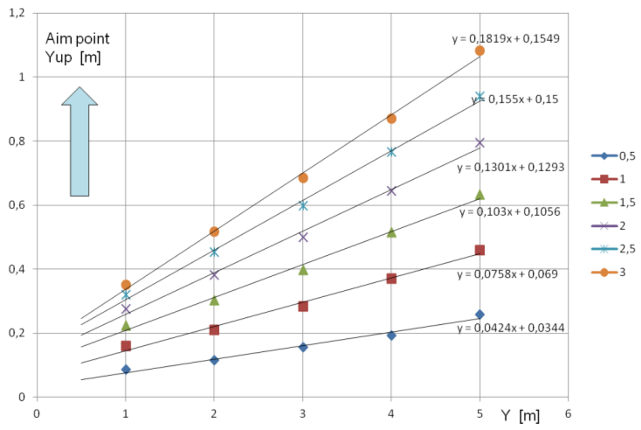

Fig. 11 - Uphill putts with "0,5m putt strategy" for different slant angles Θ of the green.

The aim point is nearly linear with the slant angle and distance in y-direction. The following approximation will work for the daily life and is a compromise between best approximation and an easy calculation for the golfer. The constant of 11mm in below mentioned approximation was selected only for the purpose of having an "easy" formula, much similar to the downhill case.

$$Y \text{ aim point} \sim 11 \text{ mm } SF \;\; \Theta \;\; Y \qquad (5.19)$$

with	SF	= Stimpfactor of green	6
	Θ	= slant angle in degree	3
	Y	= distance in meter	4

Example: Y = 11mm 6 3 4 = 792 mm (see above)

You could easily compensate the "not exact" value of aim point by a different ball velocity. Remember figure 8 and 9 for different velocities while sinking a putt.

Nearby: a 10 cm difference in aim point of 0,8 m will result for X = 2 m and Y = − 4m in a target angle difference of 0,42 degree. Not easy to see and to realise in practice.

The following chart demonstrate a putt to the hole if you are going around at a certain distance as a function of the slant angle Θ . You see clearly the quadratic dependence for downhill case, compared with the more linear behaviour of an uphill putt. The values for aim point are varying by +/- 10 cm. By the way this chart is also using the "0,5 m to long putt strategy"

Parameter of this chart: circle with 3 m radius, Stimpfactor of green is 8; equivalent to $\mu_R = 0{,}07$

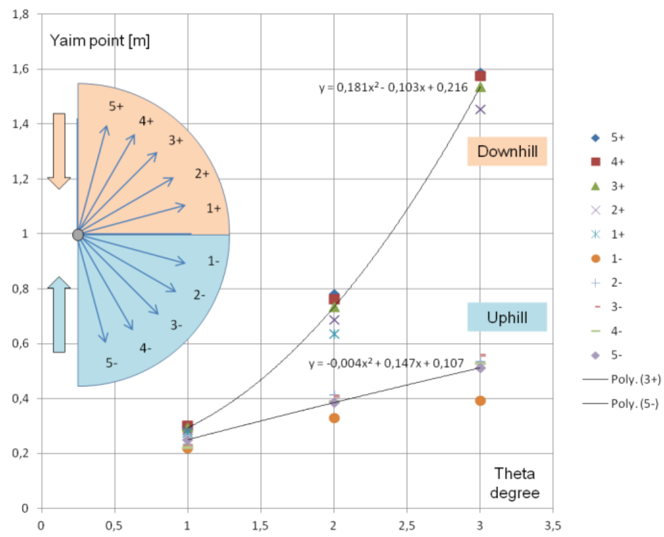

Fig. 12 - Putting on a constant circle around the hole for different slant angles.

A final example demonstrate sinking a ball for two extremes with

1. minimum speed, just to reach the hole on the upper edge or with
2. maximum speed of e.g. 1,5 m/sec direct through the centre of the hole (the absolute maximum speed has been calculated by A.R. Penner and is app. 1,65 m/sec - see ref. 2)

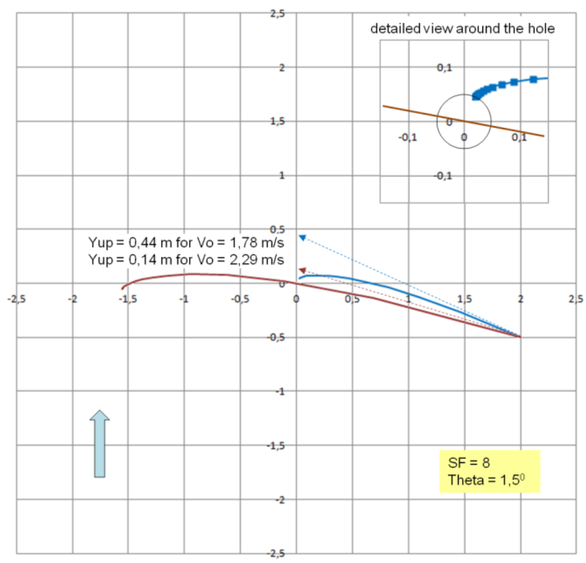

Fig. 13 - Uphill putt from the side for minimum and maximum speed

Every golfer has in direction and speed acceptable ranges for a successful putt.

The following graphics indicates the range of aim points on the y-axis for the cases velocity = min and max as above described.

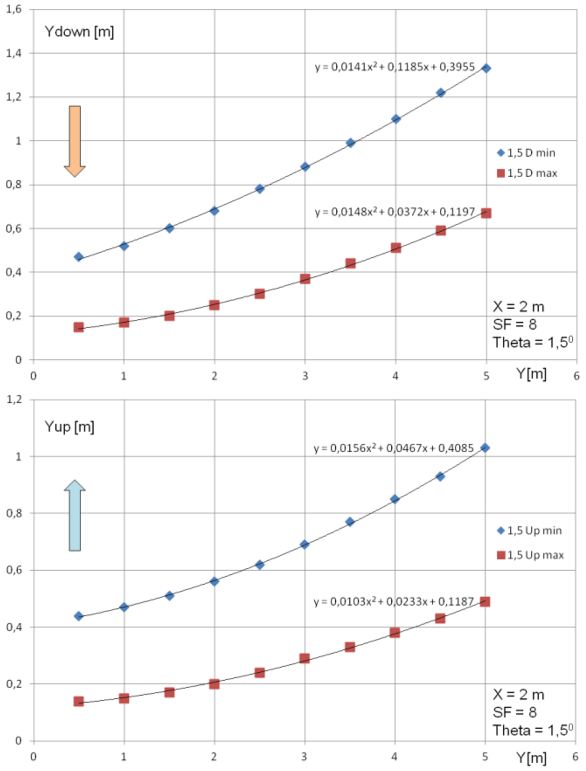

Fig. 14 - Range of aim point on the y-axis for downhill and uphill putt.

Everybody has enough room for maneuvering in aim point and velocity to sink the ball.

VI: Comparison with other solutions

Robert D. Grober of Yale university has published "The geometry of putting on a planar surface" the following graphics, based on a very detailed numerical analysis.

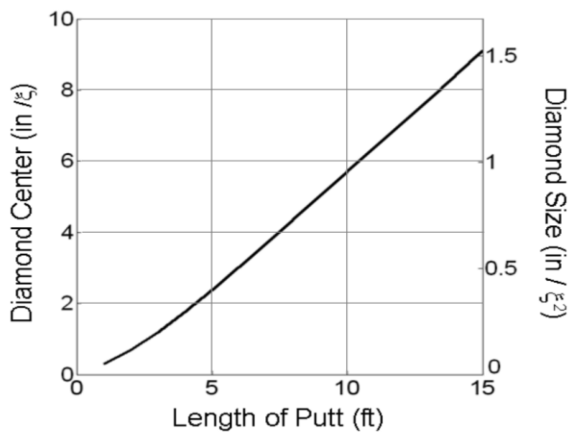

Fig. 15 - Aim point "Diamond center" vs putt length

A linear approximation of this chart is

Y aim point ~ 5 mm R SF Θ	(6.1)

with SF = Stimpfactor of green
Θ = slant angle in percent !
R = radius distance in meter

Another solution was presented by "Aim Point" guru Jamie Donaldson in his publication "The art of green reading".

Fig. 16 - The aiming angle ϑ will be realised by the numbers of fingers. The finger diameter of app. 2 cm corresponds with an arm length of 60 cm to an angle of ~ 2 degree (arctan 2/60)

You get easily the following relationships:

$$\tan\varphi = r/R$$

$$\tan\vartheta = b/R$$

$$\tan\alpha = Y_0/R$$

$$\tan(\alpha+\vartheta+\varphi) = \frac{Y_0 + Y_{up}}{X_0}$$

$$\tan(\alpha-\vartheta-\varphi) = \frac{Y_0 - Y_{down}}{X_0}$$

with $X_0 = \sqrt{R^2 - {Y_0}^2}$

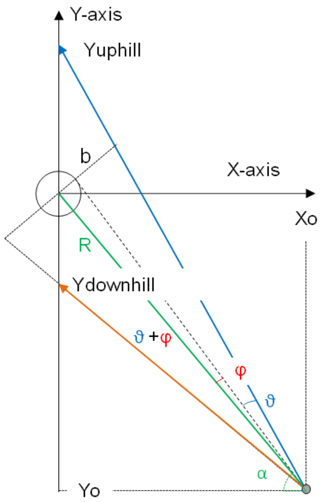

Fig. 17 - aim point geometry

Up:	$Y_{up} = Y_0 + \sqrt{R^2 - {Y_0}^2}\ \tan(\alpha + (\vartheta+\varphi))$	(6.2)
Down:	$Y_{do} = Y_0 - \sqrt{R^2 - {Y_0}^2}\ \tan(\alpha - (\vartheta+\varphi))$	(6.3)

These simple relationships describe the different aim points for uphill and downhill just by some angles. The "secret" in itself are the values for ϑ. A relationship with the green speed/ the Stimpfactor is not given. The aiming angle ϑ has an unknown connection to the slant angle Θ.

But obviously this art of green reading is well accepted by professional golfers. May be due to the fact that greens on PGA tours are all over the world very fast.

The following diagrams demonstrate the performance of this equations for uphill and downhill.

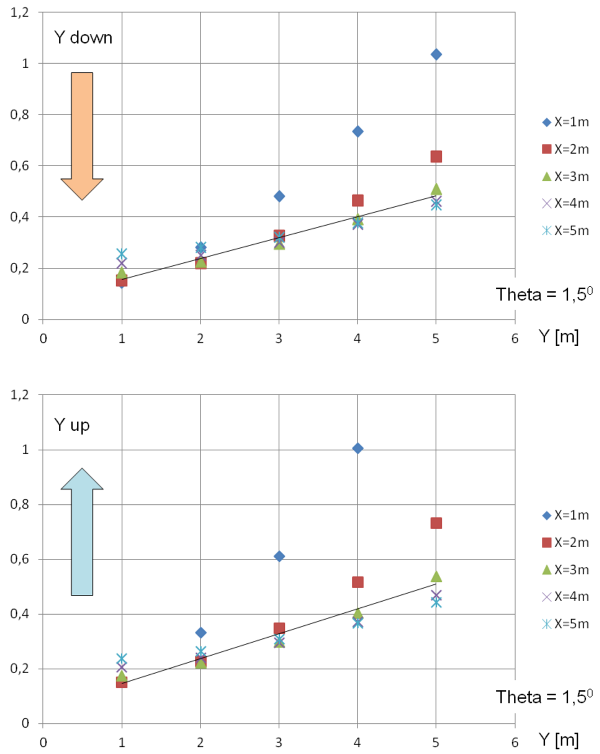

Fig. 18 - Aim Point method by Donaldson

A comparison with the exact solution in para. 5 demonstrates an acceptable behaviour. Nevertheless the results could be improved by using a multiplied slant angle relationship to the aiming angle like $\vartheta \sim \theta^k$.

We are now in a position to compare all available and published putt strategies. The solutions are:

1. The exact solution of Newton's equation for a golf ball on a slanted green with friction as in para. 5. We will use always the "0,5 m putt strategy" - that means in case of missing the hole the ball will stop at 0,5 m at the rear of the hole. We will use the min-max values for sinking a ball to demonstrate the corridor of available values for the aim point.
2. Diamond center target strategy by Grober for uphill case
3. Aim point strategy by Donaldson

The solution by Donaldson, which is really easy to practice, is surprisingly well located for uphill cases. For downhill cases you have to adopt the equation. The main problem of Donaldson approach is the missing (or not published) relationship to the different green speed/ stimpfactor.

Nevertheless an interesting result !

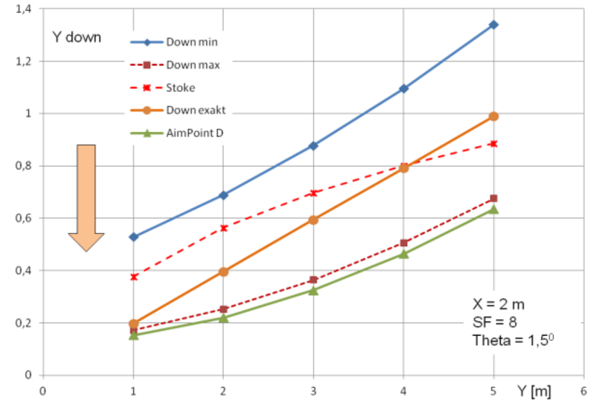

Fig. 19 - Comparison of different solutions for downhill

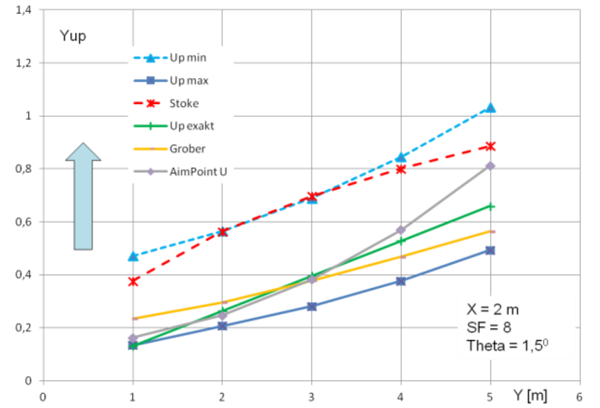

Fig. 20 - Comparison of different solutions for uphill

Nearby: in fig. 19 and 20 is in addition integrated a calculation with friction forces for the so called "Stoke" case where the friction force is dependent on the ball speed.

VII: The stepped green

The exact solution of para 5 could be used to model the very critical and difficult stepped green.

Fig. 21 - Illustration of a smooth stepped green

The solution for slanted planes will the basis of building stepped greens just by dividing the original smooth stepped function in a "digital" curve with various slant angles.

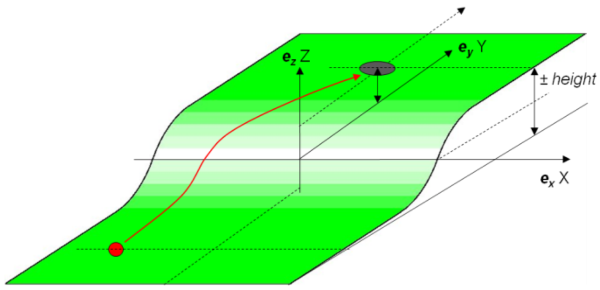

Fig. 22 - A stepped green with hole on the upper part, golf ball will start from the lower part

There are several functions available which are a good starting point for modelling. We have used the inverse tangents function with a parameter "Height" H and a "height gradient" HG for maximum flexibility.

$$Z = \mathrm{H}\,\frac{2}{\pi}\,\tan^{-1}(HG * Y) \tag{7.1}$$

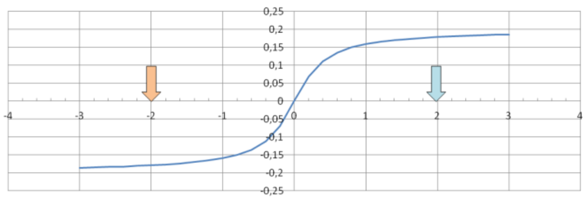

Fig. 23 - Modelling of stepped green with Total Height of ± 0,2 m, start position on left side, hole position 2 m

The gradient of the green function is the derivative

$$\frac{dZ}{dY} = \mathrm{H}\,\frac{2}{\pi}\,HG\,\frac{1}{1+(HG*Y)^2} \tag{7.2}$$

The friction coefficient of rolling μ_R (equivalent to the Stimpfactor SF) is acting as a gradient and sets the limit on Y, where the golf ball is able to stay. In the middle of the invtan function the gradient will be to strong, so that no ball will be able to stay there forever.

$$Y_{limit} = \pm\frac{1}{HG}\sqrt{\frac{2\,H\,HG}{\pi\,\mu_R} - 1} \tag{7.3}$$

Example: H = ± 0,2 m, HG = 3, Stimpfactor = 8 ($\mu_R = 0{,}07$) leads to Y_{limit} = ± 0,704 m and a median gradient of 8,9 % between -2 m < Y < 2 m

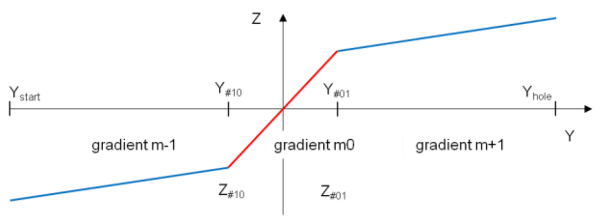

Fig. 24 - linearization of any smooth function

The real function will be modelled by several linear function Z_i

$$Z_i = m_i Y + n_i$$

Two functions Z_i and Z_{i+1} has a point of intersection at

$$Y\# = -\frac{n_i - n_{i+1}}{m_i - m_{i+1}} \quad \text{with} \quad Z\# = m_i Y\# + n_i$$

The different values of the straight lines will be set by

- the start of the putt on the left side (for uphill) for calculating the start gradient m_- and constant n_-
- the maximum gradient m_0 at the point of origin
- end of putt at the hole with gradient m_+ and constant n_+

See the following examples just for illustration.

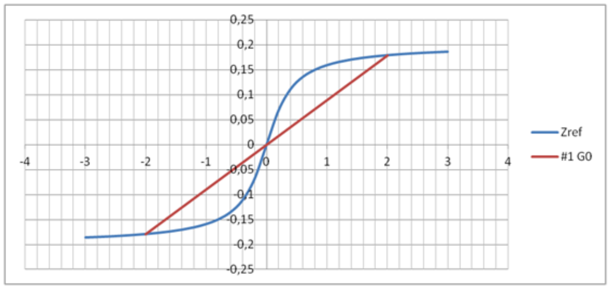

Fig. 25 - One step

1	mi	ni	Y#	Z#	m#	n#
0	8,95%	0				

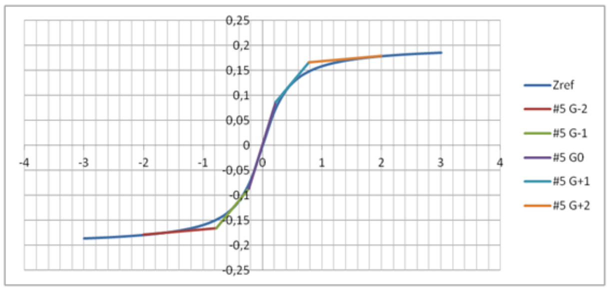

Fig. 26 - Five steps

5	mi	ni	Y#	Z#	m#	n#
-2	1,0%	-0,158	-0,777	-0,148	0,059	-0,102
-1	14,5%	-0,054	-0,227	-0,076	0,261	-0,017
0	38,2%	0	0,227	0,076	0,261	0,017
1	14,5%	0,054	0,777	0,148	0,059	0,102
2	1,0%	0,158				

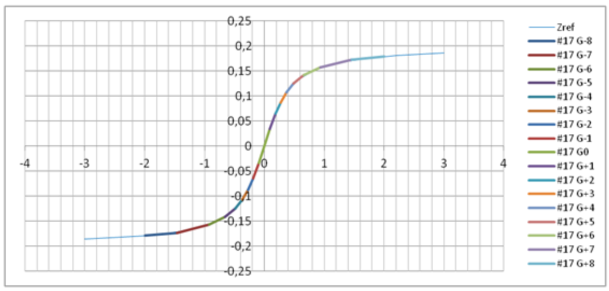

Fig. 27 - With seventeen steps every smooth function is nearly perfect modelled

The following graph demonstrate the excellent modelling performance for an uphill putt

By the way - all calculations have been performed with a seventeen step model of green which resulted in a numerical calculation error of app. 1 mm.

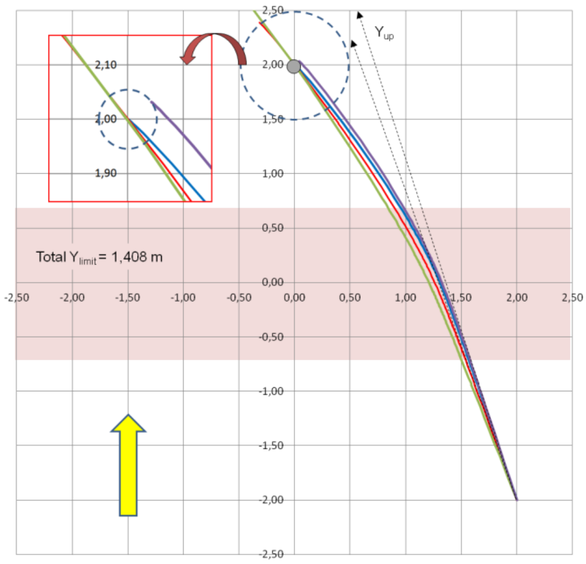

Fig. 28 - Uphill putt on a stepped green, parameters as before with H = ± 0,2 m, HG = 3, $\mu_R = 0{,}07$

You see four different putt approaches with

- velocity = 0, a direct central putt (blue)
- velocity = 0 at hole edge for maximum Y_{up}= 4,18 m (see violett curve)
- a certain start velocity so that the putt will at least stop 0,5 m at the rear edge of the hole (red)
- a maximum velocity of V = 1,5 m/sec at the hole which lead to minimum Y_{up}= 3,23 m (green)

The velocities has a span between 3,62 ... 3,91 m/sec (roughly 10 %) for an aimimg point difference of nearly one meter - quite challenging.

If you are starting from the lower platform of the green uphill you are in quite good position (from a golfer point of view).

- The target point on the Y-axis is strongly dependent of the total green height
- has surprisingly a limited dependence of the start position in Y and
- also the dependency of your own start position in X is relatively low

Please note: a target variation of ± 10 cm is equivalent to only ± 0,28 degree in angle (for Xstart = 2m, Ystart = −2 m and Ytarget = 4 m)

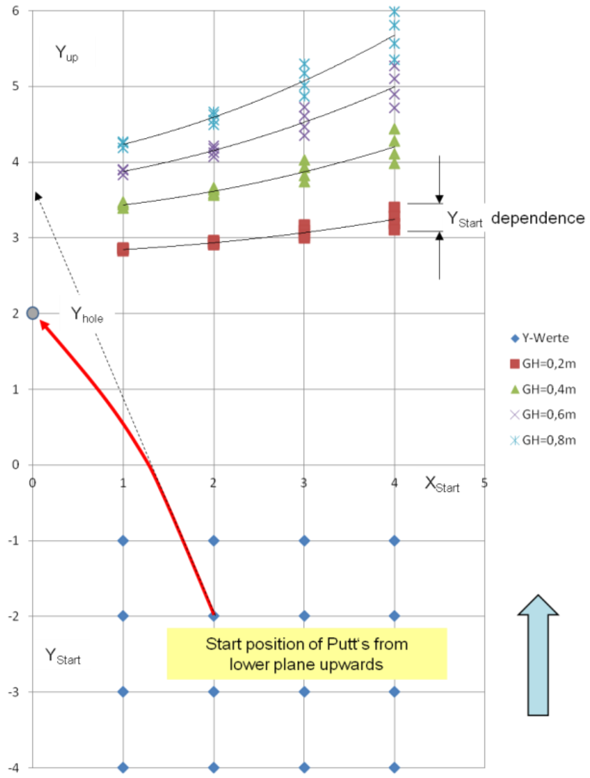

Fig. 29 - Uphill putt on a stepped green with aiming point for different total step height and start positions, friction coefficient always $\mu_R = 0{,}07$

Variation of the other variables like height gradient HG, friction coefficient and hole distance from the center has to be taken into account by some Correction Factors CF as follow:

1. Height gradient

   variation of ΔHG will have a direct influence on Y_{limit} (see 7.3) and will modify the aiming point proportional

   CF= 0,2 m ΔHG

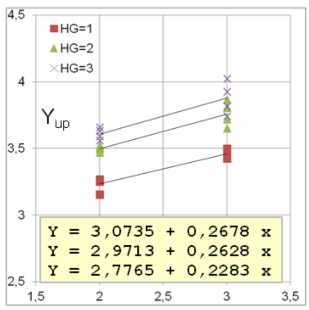

Fig. 30 - Height gradient

2. Friction coeefficient

   with an increasing friction force the Stimpfactor SF will be reduced by ΔSF with a similar influence to Y_{limit} and therefore

   CF = 0,15 ΔSF

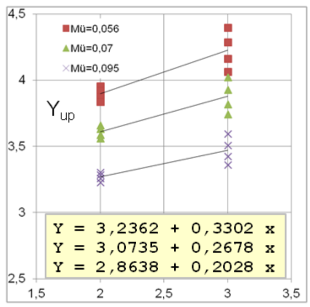

Fig. 31 - Stimpfactor

3. Hole distance

The distance to the coordinate centre is a linear relationship to the aiming point. The error is in the order of 10 cm, therefore a correction factor for distance variation is not neccessary.

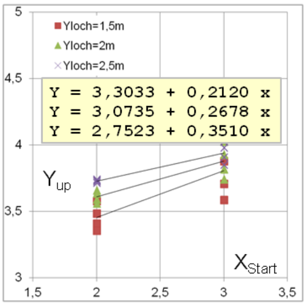

Fig. 32 - hole variation

The following approximation describe the siming point as a function of

$$Y_{up} = Y_{hole} + 3\ GH + 0{,}6\ GH\ X_{start} + CF \qquad (7.4)$$

Total Green Height GH/ X_{start} and Y_{hole} in meter, height gradient = 3 and friction coefficient is 0,07 equivalent to Stimpfactor = 8

The downhill putt is from a golfers point of view much more difficult. If you are missing the "right" corridor you will miss the hole and the ball will be accelerated by gravity force.

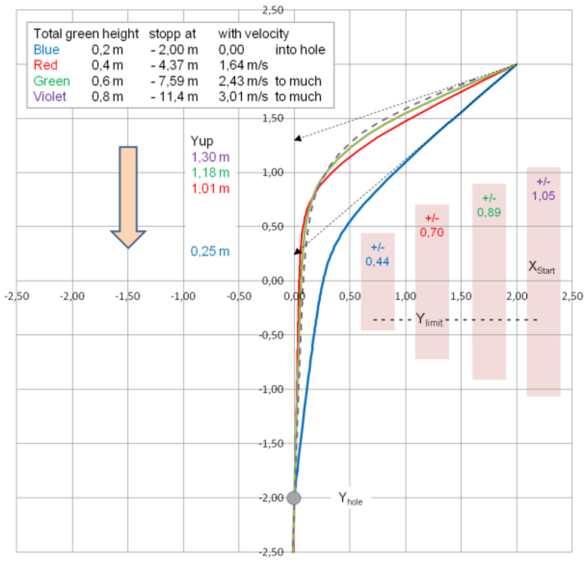

Fig. 33 - Downhill putt on various stepped green with different Y_{limit} values. Please note: the blue step of just 20 cm is not a real step, it is more acting as a normal downhill putt on a slanted green.

Because of the strong gradient and the natural acceleration by gravity force all downhill putts have to direct to the centre. All other parameters have very limited influence. You have to direct the ball in this form that you reach with nearly minimum speed the point above the hole, the rest will be done "automatically" with the help of gravity force.

See the following chart for different start positions

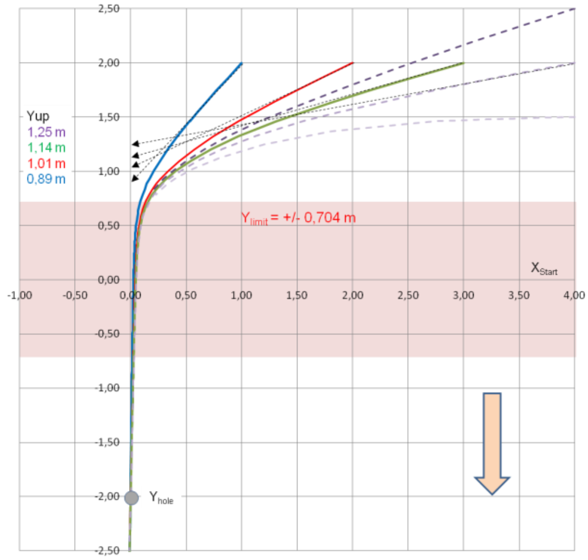

Fig. 34 - Downhill putt for different Xstart
Nearby: all putts will stop (in case of missing the hole) at Y = − 4,37 meter, independent of Xstart value

Its unbelievable - but the following equation will be a very nice approximation for the downhill aim point.

$$Y_{down} = Y_{limit} + 0{,}15\ X_{start} \qquad (7.5)$$

You have just to calculate the Y_{limit} value with 7.3

VIII: Conclusion

Based on an exact solution for the ball movement on slanted and stepped green, we have developed simple approximation for the so called "aim point" for uphill as well as downhill situations.

The intention of the author was to develop a simple "tool" for every golf player with some mathematical/ physical background. The approximations have to be easy enough to be done during the golf game. Due to the fact that beside an exact aim point a golfer need also the "correct" speed for his ball all approximations has been centred in the corridor between maximum and minimum allowable speed.

And the "optimum" speed has still to be trained by an intensive training on the putting green.

Good luck.

IX: Acknowledgement

Prof. Iván Egry from RWTH university in Aachen; Germany with his book on "Physics of Golf" was my personnel trigger to think about putting in detail. Thank you very much for your inspiration and this excellent book.

X: References

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